The Proper Time Equation and the Zamolodchikov Metric

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February 1, 2008

Abstract

The connection between the proper time equation and the Zamolodchikov metric is discussed. The connection is two-fold: First, as already known, the proper time equation is the product of the Zamolodchikov metric and the renormalization group beta function. Second, the condition that the two-point function is the Zamolodchikov metric, implies the proper time equation. We study the massless vector of the open string in detail. In the exactly calculable case of a uniform electromgnetic field strength we recover the Born-Infeld equation. We describe the systematics of the perturbative evaluation of the gauge invariant proper time equation for the massless vector field. The method is valid for non-uniform fields and gives results that are exact to all orders in derivatives. As a non trivial check, we show that in the limit of uniform fields it reproduces the lowest order Born-Infeld equation.

1 Introduction

The sigma model renormalization group approach to computations in string theory [1, 2, 3, 4], and its various generalizations, [5-25], have been very fruitful. In its original version it involves calculating various β-functions for the generalized coupling constants of the sigma model. The equations of motion that one obtains from an S-matrix calculation are proportional to the β-function, the proportionality factor being the Zamolodchikov metric[37]. The fact that there must be a proportionality factor, was inferred in [15] by calculating the β-function exactly for a constant electromagnetic field background, and showing that it could not be obtained from an action unless multiplied by a prefactor. Polyakov outlined an argument that showed that the proportionality factor is the Zamolodchikov metric [36]. In [10] this was demonstrated in detail for the tachyon.

In [10] it was also shown that an equation, called the proper time equation (which is similar to the proper time equation for a point particle[28, 29, 30, 31, 32, 33]) is in many ways easier to calculate than the β-function, especially for the tachyon and massive modes. The calculation of the various terms of the proper time equation is very similar to an S-matrix calculation in first quantized string theory. It is therefore quite easy to see that it gives the full equation of motion of the string modes (i.e. including the Zamolodchikov metric prefactor)[10]. In [10] the discussion was confined to the tachyon. More recently [26], it was applied to the massless vector field where some results are known [15, 16, 17, 18]. In particular it is known that in the limit of constant field strength, the equation of motion of the photon is that derived from the Born-Infeld action,

$$\frac{\delta}{\delta A_\mu}\int d^D X\sqrt{Det(I+F)} = \sqrt{Det(I+F)}(I-F^2)^{-1}_{\mu\nu}\partial_\rho F_{\nu\lambda}(I-F^2)^{-1}_{\lambda\rho} = 0 \tag{1.1}$$

The exact β-function for this theory in this limit is also known [15]

$$\beta_\nu = \partial_\rho F_{\nu\lambda}(I-F^2)^{-1}_{\lambda\rho} \tag{1.2}$$

The prefactor in (1.1) is thus the Zamolodchikov metric.

In [26] leading corrections to Maxwell's equation were calculated using the proper time formalism. It was shown that the zero momentum limit of these corrections agreed with the Born-Infeld result - namely that they come

from the variation of

$$TrF^4 - 1/4(TrF^2)^2 \tag{1.3}$$

Actually this was demonstrated in a gauge fixed calculation involving transverse, on-shell photons satisfying $k^2 = k.A = 0$. The gauge invariant calculation is a little more tricky and only partial results were presented - it was shown that the $(TrF^2)^2$ is present with a non-zero coefficient. This term is significant because it does not show up in the β-function calculation [15]. (See equation 1.2). The reason for this is explained in [26]. The fact that it shows up in the proper time equation is evidence that it is the complete equation.

It should also be pointed out that the proper time equation gives results that are exact to all orders in derivatives. This is unlike sigma model perturbation theory where one performs a derivative expansion. In particular the equation is valid for finite values of momenta and all the massive poles are manifest just as in an S-matrix calculation. Corresponding calculations of β-functions are much harder because one has to disentangle various subdivergences [10, 26]. One look at the β-function calculation for the Sine-Gordon theory [34] or the tachyon [6] should convince anyone of this.

In this paper we would like to set up the systematics of evaluating the gauge invariant proper time equation for the massless vector field. Our perturbation series will be in the field strength. At each order in the field strength we will write down a formal expression that is exact to all orders in the momentum of the vector field. The formal expression is very similar to the Koba-Nielsen representation. In some regions we can evaluate it exactly in terms of Gamma functions and related functions. The results, by construction, agree with S-matrix when the fields are exactly on shell. The hope is that the proper time equation is valid even when the fields are not exactly on-shell. When one is far off-shell one has to resort to other devices (perhaps as in [35]) or string field theory [38].

There are some simplifications and also some subtleties that one must be aware of when actually doing the calculation. In the Koba-Nielsen representation the integration variables satisfy $0 \le z_1 \le z_2 \le z_3 \le ... \le z_N \le 1$. We could set up the problem in the same way for the proper time equation also - and this was done in [10]. In β-function calculations, on the other hand, the range of integration is from $-\infty$ to $+\infty$. There are certain advantages to this: for instance the following identity for 2-D Green functions (easily

verified by going to momentum space) enormously simplifies the β-function calculation [15, 16]

$$\int_{-\infty}^{\infty} dz' \partial_z G(z-z') \partial_{z'} G(z'-z'') = \delta(z-z'') \tag{1.4}$$

It is crucial here that the range of integration be from $-\infty$ to $+\infty$. We will see that we can fruitfully modify the range of integration to $(-\infty, +\infty)$ in the proper time equation also. Another subtle issue is the zero momentum limit. It will turn out that in some of the calculations it is necessary to keep momenta non-zero initially and then take the limit of zero momentum. This is because many of these expressions are divergent in the zero momentum limit and there are cancellations between poles and zeros that one needs to keep track of.

One can also study the zero momentum limit directly as in [15, 16]. In this case one can use the exact Geen functions and check that the proper time equation gives the full equation. One can also separately calculate the Zamolodchikov metric and confirm earlier results.

This paper is organized as follows. In Section 2 we describe in general terms the connection between the proper time equation and the Zamolodchikov metric. In Section 3 we look at the case of a uniform electromagnetic field and obtain the full Born-Infeld equation. In Section 4 we describe gauge invariant perturbation theory for the proper time equation in the case of non-uniform electromagnetic field. In Section 5 we take the zero momentum limit of the general result and show that there is agreement with the results of Section 3. We conclude in Section 6 with a summary and some comments.

2 Proper Time Equation and the Zamolodchikov Metric

The proper time equation in its simplest form (for the tachyon) is

$$\int d^D k \{ \frac{d}{d\ln z} z^2 < V_p(z) V_k(0) > \} \bigg|_{\ln z = 0} \Phi(k) = 0. \tag{2.1}$$

The evaluation of the derivative at $\ln z = 0$ ensures that higher powers of $\ln z$ do not contribute to the equation. In (2.1) Φ is the tachyon field associated with the vertex operator V. We have indicated the momentum dependence of the vertex operator by the subscripts. The full tachyon perturbation is

$$\int dz d^D k e^{ik.X(z)} \Phi(k) \equiv \int dz d^D k V_k(z) \Phi(k). \tag{2.2}$$

We have to assume that $\Phi(k)$ is non zero only in a small region around $k^2 = 2$ in order for (2.1) to make sense. We are assuming that the fields are almost on-shell , i.e. they are marginal perturbations. This is a serious limitation in all these approaches. To get around this one needs to retain a finite cutoff on the world sheet [10, 19, 35]. We will not discuss this aspect of the problem in this paper and henceforth we will assume that all fields are almost on-shell. For massless fields this means $k^2 \approx 0$. Note that this does not mean that $k \approx 0$. Thus k can be finite (but smaller than $\frac{1}{\sqrt{\alpha'}}$)[1].

The vacuum expectation value in (2.1) is evaluated with the usual Polyakov measure but with (2.2) added as a perturbation. Thus one can evaluate (2.1) as a power series in Φ. Furthermore the range of integration is from 0 to 1. It is also important to regularize (2.1) with an ultraviolet cutoff. One simple and convenient way is to alter the limits of integration to ensure that there is a minimum spacing a between two vertex operators. This has the effect of subtracting pole terms which is exactly what one wants in an effective action [10].

It was shown in [10] that the above prescription reproduces the equation of motion. The argument is very simple: In an S-matrix calculation, because of SL(2,R) invariance, one would hold three of the vertex operators fixed - say

[1]If $k \geq \frac{1}{\sqrt{\alpha'}}$ we run into issues about non-renormalizability of the model and the inclusion of massive modes [27]

at 0, z and z_1 with $0 \le z_1 \le z$. The result of integrating $z_2, z_3, ..., z_N$ is to produce the N-particle S-matrix element multiplied by $z^{-1+\epsilon}(z-z_1)^{-1+\delta} z_1^{-1+\gamma}$, where ϵ, δ, γ are infinitesimals that vanish when the particles are exactly on-shell. In an S-matrix calculation this factor would then exactly cancel the Jacobian from SL(2,R) gauge fixing: $z(z-z_1)z_1$. However in (2.1) there is a further integration over z_1. On doing this integral and looking at the $\ln z$ deviation from $1/z^2$ and taking the limit $\epsilon = \delta = \gamma = 0$ we merely get a factor 2, which multiplies the S-matrix element. The effect of the ultraviolet regulator is as mentioned before, to subtract pole terms from the S-matrix. To show that this is also equal to the Zamolodchikov metric times the β-function requires a little more work. We refer the reader to [10, 36].

One can also generalize (2.1) by adding indices to the vertex operators and fields. These could be Lorentz indices or a general index indicating the different modes of a string. Thus we get the general proper time equation:

$$\sum_J \{ \frac{d}{d \ln(z/a)} z^2 < V_I(z) V_J(0) > \} \Big|_{\ln(z/a)=0} \Phi^J = 0. \tag{2.3}$$

This is a set of equations, one for each value of the index I. Note that, with an ultraviolet regulator in the form of a short distance cutoff, a, $\ln z$ is actually $\ln(z/a)$. Thus we are evaluating the derivative with respect to $\ln z$ at $z = a$. (This point was pursued further in [10, 19, 35] to study off-shell theories.) If one works with renormalized quantities this could be rewritten as $z = b$ where b is some renormalization scale.

The two-point function is related to the Zamolodchikov metric. Consider the following general expression for the two-point function of two VO's, both marginal (i.e. dimension 1):

$$< V_I(z) V_J(0) > = \frac{G_{IJ}}{z^2} + \frac{H_{IJ}}{z^2} \ln(z/a) + O(\ln^2(z/a)) \tag{2.4}$$

In general $G_{IJ}(\phi)$ and $H_{IJ}(\phi)$ are functions of the background fields - the coupling constants of the two dimensional theory. Here G_{IJ} is the Zamolodchikov metric. Thus

$$< V_I(a) V_J(0) > = \frac{G_{IJ}}{a^2} \tag{2.5}$$

Comparing (2.4) with (2.3) the proper time equation is

$$\sum_J H_{IJ} \Phi^J = 0 \tag{2.6}$$

Thus when the two point function is equal to the Zamolodchikov metric (upto terms of $O(\ln(z/a)^2)$) the proper time equation is satisfied. The reverse is not quite true since there is a sum over the index J. We can however say that the two point function and the Zamolodchikov metric viewed as matrices, are equal when acting on the subspace of solutions of the proper time equation. Thus acting on this subspace:

$$z^2 < V_I(z)V_J(0) >= G_{IJ} + O(\ln^2(z/a)) \tag{2.7}$$

The original motivation for the proper time equation was quite different. But we find this connection with the Zamolodchikov metric very interesting.

We can go a little further. We also know that

$$H_{IJ}\Phi^J = G_{IJ}\beta^J \tag{2.8}$$

This is just the statement made earlier that the proper time equation is the Zamolodchikov metric times the β-function. Here

$$\beta^J \equiv -\frac{d\Phi^J}{d\ln a} \tag{2.9}$$

Thus we find

$$z^2 < V_I(z)V_J(0) > \Phi^J = G_{IJ}\Phi^J + G_{IJ}\beta^J \ln(z/a) \tag{2.10}$$

If we let $z = \lambda a$, where $\lambda = 1 + \epsilon\ \ , \epsilon \approx 0$ then the RHS of (2.10) can be written as

$$G_{IJ}\Phi^J + G_{IJ}\delta\Phi^J \tag{2.11}$$

where

$$\delta\Phi^J = \beta^J \ln\lambda \tag{2.12}$$

which is the change in Φ^J under the scale change $a \rightarrow \frac{a}{\lambda}$. Thus

$$z^2 < V_I(z)V_J(0) > \Phi^J = G_{IJ}(\Phi^J + \delta\Phi^J). \tag{2.13}$$

which is nothing but the statement that the evolution of the string is really a renormalization group flow in the two dimensional theory.

The connection with the Zamolodchikov metric also suggests a connection with the background independent formalism of [22, 23]. In fact one could

try to transcribe the proper time equation into the BRST formalism. In the BRST formalism one would have to include the ghosts. The renormalization group transformation would then be replaced by a BRST transformation. This should give us an equation similar to that derived in [22, 23, 24].

This concludes our review and our discussion of the proper time equation in general terms. In subsequent sections we specialize to the case of the massless vector field in the open string.

3 Uniform Electromagnetic Field

We now turn to the massless vector field in the limiting case of (an almost) uniform field strength. This has been discussed in [15, 16, 17]. The vector field perturbation added to the Polyakov action is

$$d^D k \int dz A_\mu(k) : e^{ikX(z)} \partial_z X^\mu : \tag{3.1}$$

The regularized "effective action" [2] has been computed exactly [16, 17] and is

$$< 0 \mid 0 >_F = \sqrt{Det(I+F)} \tag{3.2}$$

The Green's function for the X - fields is known exactly in this limit: [15]:

$$\Sigma_{\mu\nu}(z-z') = (I-F^2)^{-1}_{\mu\nu} \ln(z-z') \tag{3.3}$$

where z and z' are on the boundary (the real axis).

In the proper time formalism applied to the vector field [26] the following representation for the vertex operator (3.1) was useful in obtaining covariant equations[3]

$$A_\mu(k)\partial_z X^\mu e^{ikX(z)} = \int_0^1 d\alpha \partial_z (A_\mu X^\mu e^{i\alpha kX}) + i \int_0^1 d\alpha \alpha k_{[\mu} A_{\nu]} (X^\mu \partial_z X^\nu e^{i\alpha kX}) \tag{3.4}$$

We can thus calculate the two point function:

$$\int_0^1 d\alpha\alpha \int_0^1 d\beta\beta < X^\mu \partial X^\nu(z) e^{i\alpha kX} X^\rho \partial X^\sigma(0) e^{i\beta pX} >_F \, p_{[\rho} A_{\sigma]}(p) k_{[\mu} A_{\nu]}(k) \tag{3.5}$$

The coefficient of A_σ gives the equation of motion. The subscript "F" indicates that the calculation is done with the background field A_μ in the action. We have discarded the total derivative term, since we know from gauge invariance that the final result depends only on the field strength.

[2] We have put quotation marks around the words 'effective action' because it has not really been proved that it is the effective action. It is plausible, though, because the process of regularization does subtract the massless poles, at least in the scheme used in [10]. The regularization used in computing (3.2) is the zeta function regularization.

[3] This is easily proved by integrating by parts on α in the second term. In [35], where it was first used, a rather long proof was presented. That proof involved proving an intermediate result, that is more general than is required for establishing this result.

The result is

$$\sqrt{Det(I+F)}(\Sigma^{\mu\rho}\partial_z\partial_{z'}\Sigma^{\nu\sigma}+\partial_{z'}\Sigma^{\mu\sigma}\partial_z\Sigma^{\nu\rho})p_{[\rho}A_{\sigma]}k_{[\mu}A_{\nu]} \tag{3.6}$$

Here we have set $k=p=0$ in the exponentials, since we are only interested in terms that are lowest order in momentum. Using (3.3) one extracts the coefficient of $\frac{\ln z}{z^2}A_\sigma$ to get

$$\frac{\delta S}{\delta A_\sigma} = \sqrt{Det(I+F)}(I-F^2)^{-1\sigma\nu}(I-F^2)^{-1\lambda\mu}\partial_\lambda F_{\nu\mu}=0. \tag{3.7}$$

This is the equation of motion obtained by varying $\sqrt{(det(1+F)}$ [15].

One can also calculate the Zamolodchikov metric easily

$$<\partial_z X^\mu \partial_{z'} X^\nu(0)>=\frac{1}{z^2}(I-F^2)^{-1\mu\nu}\sqrt{Det(I+F)}\equiv\frac{G^{\mu\nu}}{z^2} \tag{3.8}$$

Comparing this with (3.7) one also recovers the β-function [15]:

$$\beta^\nu=(I-F^2)^{-1\lambda\mu}\partial_\lambda F_{\nu\mu} \tag{3.9}$$

Thus we conclude from this brief discussion that, as promised, the proper time equation does give the full equation of motion. We now turn, in the next section, to the non trivial situation where the electromagnetic field is non-uniform.

4 Perturbation Theory for Non Uniform Fields

In this section we work out the details of perturbation theory for evaluating the (gauge invariant) proper time equation. Thus we have to evaluate

$$\int dkdp\frac{d}{d\ln R}R^2 < A_\mu(k)\partial_z X^\mu e^{ikX(R)}A_\nu(p)\partial_z X^\nu e^{ipX(0)} > \tag{4.1}$$

where the expectation value uses the Polyakov action perturbed by $\int dz A_\mu \partial X^\mu$. Now, in the original proper time prescription [10] the interactions were confined between 0 and z. This makes it very similar to the S-matrix calculation. However it is possible to extend the range of integration from $-\infty$ to $+\infty$ using SL(2,R) invariance. To see this, consider for e.g. a correlation involving four vertex operators:

$$< V(z_1)V(z_2)V(z_3)V(z_4) > \tag{4.2}$$

An SL(2,R) invariant measure is $\int_{-\infty}^{+\infty} dz_1 \int_{-\infty}^{z_1} dz_2 \int_{-\infty}^{z_2} dz_3 \int_{-\infty}^{z_3} dz_4$. Assuming the vertex operators are on-shell, we can fix one of them to be at R, another at 0, and a third one at z and the Jacobian is $zR(R-z)$. A common choice in S-matrix calculations is $z_1 = R \rightarrow \infty$, $z_2 = 1$ and $z_4 = 0$. The prescription in the proper time equation, on the other hand, amounts to the choice $z_1 = R,\ z_2 = z$ and $z_4 = 0$. Of course as explained in Sec 1, z is also integrated over eventually, in the proper time formalism. Thus we get

$$\int_0^R dz \int_0^z dz_3 < V(z_1 = R)V(z_2 = z)V(z_3)V(z_4 = 0) > \tag{4.3}$$

One could, instead, choose $z_2 = R,\ z_3 = z$ and $z_4 = 0$ and by SL(2,R) invariance the result would be the same. In this case z_1 would be integrated from R to ∞ (because $z_1 \geq z_2$). Thus, this would contribute a term of the form

$$\int_R^\infty dz_1 \int_0^R dz < V(z_1)V(z_2 = R)V(z_3 = z)V(z_4 = 0) > \tag{4.4}$$

Now, in contrast to an S-matrix calculation, where each vertex operator has associated with it a definite momentum, in the present case the vertex operators are of the form

$$V(z) = A_\mu(X)\partial_z X^\mu = \int dk A_\mu(k)\partial_z X^\mu e^{ikX} \tag{4.5}$$

The momentum is integrated over and consequently the vertex operators are identical and indistinguishable. Thus, by relabelling the z_i 's one sees that the only difference between (4.3) and (4.4) is the range of integration of one of the z_i ,i.e.,in (4.3) z_3 is integrated from 0 to z, whereas in (4.4) z_1 is integrated from R to ∞. In both cases z is integrated from 0 to R. We can thus take four terms that are numerically identical, namely (4.3), (4.4) and two others with ranges of integration $(-\infty, 0)$ and (z, R), add them to get a range of $(-\infty, +\infty)$. Since we have added four numerically identical terms we can divide by 4. In terms of S-matrix calculation we have added four terms with the momenta permuted amongst themselves. Thus we conclude that extending the range of integration from $-\infty$ to $+\infty$ has the effect of doing an S-matrix calculation and symmetrizing on the external momenta. Note that the range of integration of the variable z is always from 0 to R.

The above argument depends crucially on SL(2,R) invariance. Thus we have to assume that the vertex operators are physical and of dimension one. For the massless vector this means $k^2 = k.A = 0$. The equations we derive are thus *a priori* valid only in an infinitesimal region of momentum space around this. Any extrapolation beyond this will require *a posteriori* justification based on the final result. This is true, of course, for any off-shell extrapolation.

As mentioned in the introduction there are some advantages to thus extending the range of integartion. One can, for instance, use (1.4). This is useful in the zero momentum limit. In the case of abelian vectors there is another advantage. The effective action is supposed to have the massless poles subtracted out. Usually this is taken care of by regularizing the integrals. In the abelian case, however, since there is no (bare) three-photon interaction, the vector-vector scattering element does not have massless poles at all. To explicitly see that the poles are absent, one has to symmetrize on the external momenta [26]. This is a little tedious in an actual calculation. On the other hand when the range of integration is extended from $-\infty$ to $+\infty$ one sees this cancellation very easily. This will be demonstrated explicitly in the next section when we look at the zero momentum limit.

We now derive the general expression. To do this we replace $A_\mu \partial_z X e^{ikX}$

by (3.4). Thus we have to calculate

$$\int_0^1 d\alpha\alpha \int_0^1 d\beta\beta \frac{d}{d\ln R} R^2 < k^{[\mu} A(k)^{\nu]} X^\mu \partial_{z_1} X^\nu e^{i\alpha k X(R)} p^{[\rho} A(p)^{\sigma]} X^\rho \partial_{z_2} X^\sigma e^{i\beta p X(0)} > \tag{4.6}$$

One inserts some number of vertex operators of the form

$$\int_0^1 d\gamma\gamma q^{[\mu} A(q)^{\nu]} X^\mu \partial_z X^\nu e^{i\gamma q X(z)}. \tag{4.7}$$

and integrates all of them from $-\infty$ to $+\infty$, except for one, which is integrated from 0 to R. If one regulates the integrals by imposing that the distance of closest approach between two vertex operators is $a \neq 0$, then one finds that the on-shell poles are subtracted. [4]

To illustrate all this we turn to the leading correction to Maxwell's equation, involving two insertions of (4.7) in (4.6). Thus consider

$$\int dk\, dp\, dq\, dl\, F_{\mu_1\nu_1}(k) F_{\mu_2\nu_2}(p) F_{\mu_3\nu_3}(q) F_{\mu_4\nu_4}(l) \int d\alpha d\beta d\gamma d\delta \alpha\beta\gamma\delta$$

$$X^{\mu_1} \partial_{z_1} X^{\nu_1} e^{i\alpha k X(z_1)} X^{\mu_2} \partial_{z_2} X^{\nu_2} e^{i\beta p X(z_2)} X^{\mu_3} \partial_{z_3} X^{\nu_3} e^{i\gamma q X(z_3)} X^{\mu_4} \partial_{z_4} X^{\nu_4} e^{i\delta l X(0)} \tag{4.8}$$

There are two types of contractions that one can make in which the $X^{\mu_i} \partial X^{\nu_i}$ are contracted amongst themselves - one results in $Tr(F^4)$ and the other $(TrF^2)^2$). In this paper we will consider only these . Let us consider these two in turn.

a) TrF^4**:**

There are three kinds of contractions:

$$i)\ \frac{1}{z_1 - z_2} \frac{1}{z_2 - z_3} \frac{1}{z_3 - z_4} \frac{1}{z_4 - z_1}$$

$$ii)\ \frac{1}{z_1 - z_3} \frac{1}{z_3 - z_2} \frac{1}{z_2 - z_4} \frac{1}{z_4 - z_1}$$

$$iii)\ \frac{1}{z_1 - z_2} \frac{1}{z_2 - z_4} \frac{1}{z_4 - z_3} \frac{1}{z_3 - z_1} \tag{4.9}$$

[4] As mentioned earlier, in the abelian case this is unnecessary.

There are 2^4 terms of each type that are obtained by interchanging μ_i with ν_i in each contraction. Each of these are multiplied by the factor M defined below:

$$M = \mid z_1 - z_2 \mid^{\alpha\beta k.p} \mid z_2 - z_3 \mid^{\gamma\beta q.p} \mid z_3 - z_4 \mid^{\gamma\delta q.l} \tag{4.10}$$

$$\mid z_1 - z_3 \mid^{\alpha\gamma k.q} \mid z_2 - z_4 \mid^{\delta\beta p.l} \mid z_1 - z_4 \mid^{\alpha\delta k.l}$$

Since z_1 and z_4 are fixed there is really nothing to distinguish between them. Thus (i) and (ii) (of (4.9)) are identical.

b)$(\mathbf{TrF^2})^\mathbf{2}$:

Again there are three kinds of contractions.

$$i)\ [\frac{1 + \ln(z_1 - z_2)}{(z_1 - z_2)^2}][\frac{1 + \ln(z_3 - z_4)}{(z_3 - z_4)^2}]$$

$$ii)\ [\frac{1 + \ln(z_1 - z_3)}{(z_1 - z_3)^2}][\frac{1 + \ln(z_2 - z_4)}{(z_2 - z_4)^2}]$$

$$iii)\ [\frac{1 + \ln(z_1 - z_4)}{(z_1 - z_4)^2}][\frac{1 + \ln(z_2 - z_3)}{(z_2 - z_3)^2}] \tag{4.11}$$

Each of these is also multiplied by the factor M. Once again (i) and (ii) give identical results due to the indistinguishability of z_1 and z_4.

We now turn to the actual evaluation of the integrals. We will only do a few representative examples. In doing the integrals it is important to keep in mind that it is the absolute value $\mid z_i - z_j \mid$ that occurs in the argument of the logarithms and in M, whereas in the denominator of (4.9) it is $(z_1 - z_j)$ - *without* the absolute value - which can be positive or negative, depending on whether $z_i \geq$ or $\leq z_j$.

Let us consider a typical term in a)(i):

$$\int_0^{z_1} dz_2 \int_{-\infty}^{+\infty} dz_3 \mid z_1 - z_2 \mid^{k.p} (z_1 - z_2)^{-1} \mid z_2 - z_3 \mid^{q.p} (z_2 - z_3)^{-1} \mid z_3 - z_4 \mid^{q.l} (z_3 - z_4)^{-1}$$

$$\mid z_1 - z_3 \mid^{k.q} \mid z_2 - z_4 \mid^{p.l} \mid z_1 - z_4 \mid^{k.l} (z_1 - z_4)^{-1} \tag{4.12}$$

We have used k for αk, p for βp etc for notational simplicity. The factors $\alpha, \beta...$ will have to be restored at the end. We have further set $z_4 = 0$. These

integrals can be expressed in terms of Beta functions and hypergeometric functions. For simplicity let us choose a region of momentum space where

$$p.l = k.q = 0 \tag{4.13}$$

This is over and above the mass shell constraint

$$k^2 = p^2 = q^2 = l^2 = 0 \tag{4.14}$$

The two,(4.13) and (4.14), imply

$$k.p = q.l = -p.q = -k.l \tag{4.15}$$

Note, however, that $k.p, p.q...$ are not required to be small. In this sense we are going beyond the usual sigma model perturbation theory, where, because a derivative expansion is being performed, we are forced to have $k \approx 0$, and not just $k^2 \approx 0$.

(4.12) can be rewritten as:

$$z_1^{-2+k.p+p.q+k.l+q.l} \int_0^1 dz_2 (1-z_2)^{-1+k.p} z_2^{-1+p.q+q.l} \int_{-\infty}^{+\infty} dz_3 (1-z_3)^{-1+p.q} z_3^{-1+q.l} \tag{4.16}$$

where we have performed the usual rescaling of z_2 and z_3. The integrals can be easily done (see Appendix A) and the result is

$$z_1^{-2+k.p+p.q+k.l+q.l} B(k.p, p.q+q.l) \tag{4.17}$$

$$[-B(1-p.q-q.l, p.q) - B(1-p.q-q.l, q.l) + B(p.q, q.l)]$$

The logarithmic deviations from the canonical $\frac{1}{z^2}$ scaling is proportional to $(k.p+k.l+p.q+q.l)$ which, by(4.15), is zero. However, the first Beta function has a pole $\frac{1}{p.q+q.l}$. Thus we get $(\frac{k.p+k.l}{p.q+q.l}+1)$. Since $k.p = q.l$ and $k.l = p.q$, we actually get a non-zero result, namely 2. Thus the final answer is

$$\int dk dp dq dl F_{\mu_1\nu_1}(k) F_{\nu_1\nu_2}(p) F_{\nu_2\nu_3}(q) F_{\nu_3\mu_1}(l)$$

$$\int d\alpha d\beta d\gamma d\delta \alpha\beta\gamma\delta \tag{4.18}$$

$$2[-B(1-p.q-q.l, p.q) - B(1-p.q-q.l, q.l) + B(p.q, q.l)].$$

As mentioned earlier, one should really regularize these integrals and thus subtract any massless poles that might be present in (4.18). However it is easy to see using an expansion of the Beta function (Appendix B) that the would be poles, in fact, cancel. The leading term is $-3(p.q+q.l)\zeta(2)$ (which is actually zero if we use (4.15)). The final answer (4.18) is valid for finite values of momenta as long as the restriction (4.13) is satisfied. Furthermore it is assumed that (4.14) is also approximately satisfied ((4.15) is then automatic). Note, however, that (4.14) was never used in evaluating the integral. It was only used to motivate the proper time equation. In principle, one can adopt the viewpoint that the equation is correct even when the fields are off-shell. In that case the only restriction is (4.13).

One can proceed to evaluate in a similar manner all the terms in (4.9) and (4.11). We will write down the results for a couple more of the terms, since they will be used in the next section for extracting the zero momentum limit. Let us consider two more terms in ai)(4.9).

$$\frac{1}{(z_1-z_2)^2}\ln(z_2-z_3)\frac{1}{(z_3-z_4)}\frac{1}{(z_4-z_1)} \tag{4.19}$$

$$\frac{1}{(z_1-z_2)^2}\ln(z_4-z_1)\frac{1}{(z_2-z_3)}\frac{1}{(z_3-z_4)} \tag{4.20}$$

It is understood that they are to be multiplied by M (4.10). Consider (4.19) first. We replace the logarithm by $(z_2-z_3)^\mu$ and at the end we will pick the term that is linear in μ. The result of doing the integral is $(z_4=0)$:

$$z_1^{-2+k.p+p.q+k.l+q.l+\mu}\{B(-1+k.p,1+\mu+p.q+q.l) \tag{4.21}$$

$$[-B(-q.l-p.q-\mu,q.l)+B(1+\mu+p.q,q.l)+B(-q.l-p.q-\mu,1+\mu+p.q)]\}$$

It is easy to see that, when (4.15) is satisfied, the coefficient of $\mu\ln z_1$ is the term in curly brackets with μ set to 0. Once again it can be checked that the massless poles inside the square brackets cancel. The appearance of a pole in the first factor is also deceptive because it cancels against a zero. Here, and later, we have omitted writing down explicitly all the accompanying field strength factors and the integrals over α,β... etc.

Next, consider (4.20) We get $(z_4=0)$:

$$z_1^{-2}\ln z_1\{B(-1+k.p,p.q+q.l) \tag{4.22}$$

$$[-B(1-p.q-q.l,p.q)-B(1-p.q-q.l,q.l)+B(p.q,q.l)]\}$$

The expression in curly brackets is the answer. Once again regularization is unnecessary - the pole terms cancel. We will see this explicitly when we study the zero momentum limit in the next section.

Let us now turn to terms that contribute to $(TrF^2)^2$ given in (4.11).

bi)

$$[\frac{1+\ln(z_1-z_2)}{(z_1-z_2)^2}][\frac{1+\ln(z_3-z_4)}{(z_3-z_4)^2}]\times M \tag{4.23}$$

We consider the integral in the same approximation (4.13).

$$\int_0^{z_1} dz_2 \int_{-\infty}^{+\infty} dz_3 (z_1-z_2)^{-2+k.p+\mu}(z_2-z_3)^{p.q}(z_3)^{-2+q.l+\nu}(z_1)^{k.l} \tag{4.24}$$

The answer is given by the coefficients of $1,\mu,\nu$ and $\mu\nu$. The result for the coefficient of $\ln z_1$ is:

$$(\mu+\nu)B(-1+k.p+\mu,p.q+q.l+\nu)$$

$$[B(1-p.q-q.l-\nu,-1+q.l+\nu)+B(1-p.q-q.l-\nu,1+p.q) \\ +B(-1+q.l+\nu,1+p.q)] \tag{4.25}$$

One has to expand the Beta function in powers of μ,ν to get the final answer.

biii)

$$[\frac{1+\ln(z_1-z_4)}{(z_1-z_4)^2}][\frac{1+\ln(z_2-z_3)}{(z_2-z_3)^2}]\times M \tag{4.26}$$

Using a different approximation

$$p.l=k.q=-q.l=-k.p \tag{4.27}$$

we get

$$\int_0^{z_1} dz_2 \int_{-\infty}^{+\infty} dz_3 (z_1-z_2)^{k.p}(z_2-z_3)^{-2+p.q+\mu}(z_3)^{q.l}(z_1)^{k.l} \tag{4.28}$$

We have introduced, as before, μ for the logarithm. The result is

$$z_1^{-2+k.p+p.q+k.l+q.l+\mu}(1+\ln z_1)\{B(1+k.p,p.q+q.l+\mu) \tag{4.29}$$

$$[B(1-p.q-q.l-\mu,1+q.l)+B(1-p.q-q.l-\mu,-1+p.q+\mu)+B(1+q.l,-1+p.q+\mu)]\}$$

One has to extract the coefficient of $\ln z_1$ from the above, keeping terms independent of μ, and also linear in μ. We will not do this here since it is tedious and unilluminating. The low energy limit is worked out in the next section.

Thus the main results of this section are, (4.18),(4.21), (4.22),(4.25) and (4.29), which are some representative coefficients of the form TrF^4 and $(TrF^2)^2$ to Maxwell's action. In the region of momentum space satisfying (4.13), these are exact to all orders in derivatives and are therefore valid for finite values of momenta. To complete the calculation one has to pick out linear and bilinear terms in μ, ν. This can easily be done by expanding the Beta function in a power series. Alternatively they can be expressed in terms of Ψ-functions defined as [41]

$$\Psi(x) = \frac{d}{dx} \ln \Gamma(x) \tag{4.30}$$

Thus, for instance, to pick the piece linear in μ in $B(x+\mu, y+\nu)$ one can use:

$$\left. \frac{dB}{d\mu} \right|_{\mu=0} = B(x,y)[\Psi(x) - \Psi(x+y)] \tag{4.31}$$

Finally one has to perform the $\alpha, \beta, \gamma, \delta$ integrals - these are the parameters used in (3.4). In calculating correlation functions of the type in (4.8), after Wick contraction of the $X^{\mu_i}\partial_i X^{\nu_i}$ amongst themselves, one is left with a vacuum expectation value of four exponentials. Momentum conservation gives a constraint

$$\alpha k^\mu + \beta p^\mu + \gamma q^\mu + \delta l^\mu = 0 \tag{4.32}$$

Thus, (4.15) holds even when the parameters $\alpha, \beta...$ are included :

$$\alpha k.\beta p = \gamma q.\delta l$$

We can therefore replace δl^μ everywhere by $-(\alpha k^\mu + \beta p^\mu + \gamma q^\mu)$. The integral $\int d^D l$ can be replaced by $\int d^D(\delta l)$ (which is 1 because of the momentum conservation delta function) multiplied by δ^{-D}. The integral over the parameter δ is then an overall constant (infinite) factor that is common to every term and can be set to 1.

To summarize this section, we have described a well defined prescription for a (covariant) perturbative evaluation of the proper time equation.

5 Zero Momentum Limit

The purpose of this section is to consider the zero momentum limit of the results of the previous section and compare it with known results[15, 16, 17]. This will provide a non trivial check on the details of the perturbation scheme presented in the last section. It also serves to bring out certain subtleties in the process of taking the zero momentum limits in these kinds of calculations that can be a bit perplexing.

If the zero momentum limit is taken directly we run into problems. Taking the zero momentum limit is equivalent to setting the factor $M = 1$ (4.10). If we consider the original proper time prescription of having a range of integration for z_i between z_{i-1} and z_{i+1} we run into integrals of the type

$$\int_u^z dw \frac{1}{z-w}\frac{1}{w-u} \tag{5.1}$$

This is of course divergent at both ends, which is symptomatic of the infrared divergence due to massless poles $\frac{1}{p^2}$, as $p \to 0$. In the S-matrix calculation (in the abelian case), these poles cancel when we symmetrize on the external momenta. We have already seen that this is equivalent to extending the range of integration from $-\infty$ to $+\infty$. Thus we get

$$\int_{-\infty}^{+\infty} dw \frac{1}{z-w}\frac{1}{w-u} \tag{5.2}$$

If we introduce infrared and ultraviolet regulators, we end up with expressions of the type $\ln(R/a) - \ln(R/a) = 0$! Actually the cure is obvious, because what we really have is $\ln(1) = 0, 2\pi i, ...$Thus we need to be very careful about the $i\epsilon$ prescription in (5.2). In fact we should use the principal value prescription (this can be derived from the momentum representation, see Appendix C), and we get

$$1/4 \int_{-\infty}^{+\infty} dw [\frac{1}{z-w-i\epsilon} + \frac{1}{z-w+i\epsilon}][\frac{1}{w-u-i\epsilon} + \frac{1}{w-u+i\epsilon}] = -\pi^2 \delta(z-u) \tag{5.3}$$

This agrees with (1.4). The moral of this simple calculation is that one must be careful in regulating expressions like (5.1) or (5.2). One way of regulating (5.2) is to introduce $i\epsilon$ as above. Another way is to restore the factor M of (4.10) and take the limit $p^\mu \to 0$ at the end. The β-function calculation [15]

uses (5.3). This works well for the TrF^4 term but not for the $(TrF^2)^2$ term as we see below. Thus, consider the terms of (4.9). We have the result (4.18) for (ai). Expanding the Beta functions in powers of momenta one finds for the expression in square brackets (see Appendix B):

$$[B(p.q,q.l) - B(1-p.q-q.l,p.q) - B(1-p.q-q.l,q.l)] = -3(p.q+q.l)\zeta(2) \tag{5.4}$$

The pole terms have cancelled. (5.4) vanishes if we use (4.15), but even otherwise it vanishes in the zero momentum limit.

Similarly (4.21) gives for the expression in square brackets $-3(\mu+p.q)\zeta(2)$, and expanding the prefactor also, we get for the coefficient of μ,

$$\frac{1}{k.p-1}\frac{1}{k.p}(k.p+p.q+q.l)[1-k.p(p.q+q.l)\zeta(2)][-3(p.q)\zeta(2)] \tag{5.5}$$

Clearly this also vanishes in the zero momentum limit.

Finally, we turn to (4.22). We get:

$$[\frac{p.q+q.l+k.p-1}{k.p-1}][\frac{1}{p.q+q.l}+\frac{1}{k.p}](-3)(p.q+q.l)\zeta(2) \tag{5.6}$$

Although $p.q+q.l \approx 0$, the pole cancels against the zero and we get

$$-3\zeta(2) = -\frac{\pi^2}{2} \tag{5.7}$$

Thus of the three terms considered (all from (ai)), only the last one, (4.20), containing a $\ln \mid z_1 - z_4 \mid$ factor, contributes to the zero momentum limit.

If we use (5.3) this is obvious:

$$\begin{aligned} &\int_0^{z_1} dz_2 \int_{-\infty}^{\infty} dz_3 \frac{1}{z_1-z_2}\frac{1}{z_2-z_3}\frac{1}{z_3-z_4}\frac{1}{z_4-z_1} \\ &= -\pi^2 \int_0^{z_1} dz_2 \frac{1}{z_1-z_2}\frac{1}{z_4-z_1}\delta(z_2) \\ &= \frac{\pi^2}{2}\frac{1}{z_1^2} \end{aligned} \tag{5.8}$$

The delta function contributes only 1/2, because we only integrate between 0 and z_1. Evidently this does not have a $\ln z_1$ dependence, hence it does not contribute to the proper time equation.

(4.20) is

$$\int_0^{z_1} dz_2 \int_{-\infty}^{\infty} dz_3 \frac{1}{(z_1-z_2)^2}\frac{1}{z_2-z_3}\frac{1}{z_3-z_4} \ln \mid z_4 - z_1 \mid$$

$$= -\pi^2 \int_0^{z_1} dz_2 \frac{1}{(z_1-z_2)^2}\delta(z_2)\ln z_1$$

$$= \frac{-\pi^2}{2}\frac{1}{z_1^2}\ln z_1 \tag{5.9}$$

This contributes $-\frac{\pi^2}{2}$ to the proper time equation, in agreement with (5.7). Thus of the $2^4 = 16$ terms in(4.9)(ai), only four contribute to the proper time equation. The same is obviously true for (aii). Finally it is easy to see that (aiii) does not contribute at all. Thus we get a total of (with another minus sign from the Lorentz index contraction):

$$4\pi^2 TrF^4 \tag{5.10}$$

We now turn to (b) (4.11). Consider bi). The exact result is given in (4.25). The expression in square brackets gives

$$3p.q(\nu + q.l + p.q)\zeta(2) \tag{5.11}$$

and thus the full result is

$$\ln z_1(\mu+\nu)[\frac{1}{p.q+q.l+\nu} + \frac{1}{k.p+\mu}]3p.q(\nu+q.l+p.q)\zeta(2) \tag{5.12}$$

Clearly we have a momentum independent piece:

$$3\mu\nu\frac{p.q}{k.p}\zeta(2) = -3\zeta(2) = -\frac{\pi^2}{2} \tag{5.13}$$

Similarly bii) gives $-\frac{\pi^2}{2}$.

As for biii), we have (4.29) which simplifies to

$$\frac{1}{p.q+q.l+\mu}3q.l(p.q+q.l+\mu)\zeta(2) \approx 3q.l\zeta(2) = \frac{\pi^2}{2}q.l \tag{5.14}$$

which vanishes in the zero momentum limit.

Thus as far as $(TrF^2)^2$ is concerned we get a total of

$$-\pi^2(TrF^2)^2 \tag{5.15}$$

On the other hand, if we try to do the same calculation using (5.3) we get for bi)

$$\int_0^{z_1} \frac{\ln(z_1 - z_2)}{(z_1 - z_2)^2} \int_{-\infty}^{\infty} \frac{\ln(z_3 - z_4)}{(z_3 - z_4)^2} \tag{5.16}$$

We can obviously integrate by parts on z_3 and using (1.4), get $\delta(z_4 - z_4) = \delta(0)$ - which gives a divergent answer. Actually, if we use an infrared regulator in the propagator, [5] it is easy to see that the delta function acts only on non-constant functions and is zero otherwise. We therefore get zero or infinity depending on the regularization. Thus this method, which worked very well for the TrF^4 term, gives ambiguous (wrong) answers here.

The final result, for the zero momentum limit is (combining (5.10) and(5.15)):

$$4\pi^2[TrF^4 - \frac{1}{4}(TrF^2)^2] \tag{5.17}$$

To be more precise, it is the coefficient of $A^\mu(k)$ in the above expression. This clearly agrees with the Born-Infeld results (1.3) [15, 16, 17].

[5] Use $\frac{1}{p^2+m^2}$ instead of $\frac{1}{p^2}$

6 Conclusions

In this paper we have discussed, in some detail, the proper time equation for the electromagnetic field in the open string. In Section 2 we have tried to give an overview of the proper time equation and its intimate connection with the Zamolodchikov metric. In Section 3 we illustrated this with the exactly calculable case of a uniform electromagnetic field. We showed that the proper time equation gives the full (Born-Infeld) equation of motion. In Section 4 we gave a systematic prescription for evaluating the covariant proper time equation in the general momentum dependent case. We illustrated it by calculating some of the leading corrections to Maxwell's equation. We showed that one can write down in some cases, closed form expressions, that are exact in their momentum dependence for *finite* values of momenta, rather than for infinitesimal values as in the sigma model case. Furthermore one sees the presence of massive poles. Thus the radius of convergence of the momentum expansion is manifest. Finally in Section 5 we discussed the zero momentum limit and showed that it agrees with the Born-Infeld results. This provides a non trivial check on the method.

We thus conclude that the proper time equation can, using the perturbation scheme described in this paper, be used for a systematic evaluation of the equation of motion. The equations are covariant. It has also been applied to the non-abelian case [35, 26]. We believe it can be applied to the massive modes also, but the issue of gauge invariance at the interacting level has not been addressed.

The connection with the Zamolodchikov metric is interesting. The geometric significance of this metric has not been really explored. Since the metric is not a coordinate invariant object, there must be a better, coordinate invariant version of the proper time equation. The 'coordinates', here, are the D-dimensional fields, not X^μ. Perhaps some of the results of [39, 40] can be fruitfully applied here to unearth new results in string theory.

The background independent formalism of [22, 23] uses something very similar to the Zamolodchikov metric. The equation of motion also has a strong resemblence to the proper time equation. The main difference is that BRST transformations are being used rather than renormalization group transformations. It will be interesting to understand the precise connection.

Finally, one hopes that some generalization of the techniques presented here will be applicable to the massive modes and will shed some light on the

underlying principles of string theory.

APPENDIX

A Evaluation of Integrals

The first integral in (4.16), over z_2 is just the first Beta function of (4.17). The second one can be written as

$$\underbrace{\int_{-\infty}^{0} dz_3(1-z_3)^{-1+p.q} z_3^{-1} \mid z_3 \mid^{q.l}}_{(a)} \tag{A.1}$$

$$+\underbrace{\int_{0}^{1} dz_3(1-z_3)^{-1+p.q} z_3^{-1+q.l}}_{(b)}$$

$$+\underbrace{\int_{1}^{\infty} dz_3(1-z_3)^{-1} \mid 1-z_3 \mid^{p.q} z_3^{-1+q.l}}_{(c)}$$

Note that one has to be careful about whether to use the absolute value or not.

$$(a) \;=\; -\int_{-\infty}^{0} dz_3(1-z_3)^{-1+p.q} \mid z_3 \mid^{-1+q.l} = -\int_{0}^{\infty} d \mid z_3 \mid (1+ \mid z_3 \mid)^{-1+p.q} \mid z_3 \mid^{-1+q.l} \tag{A.2}$$

Using [41]

$$\int_{u}^{\infty} (x+\beta)^{-\nu}(x-u)^{-1+\mu} dx = (u+\beta)^{\mu-\nu} B(\nu-\mu,\mu) \tag{A.3}$$

we get

$$(a) \;=\; -B(1-p.q-q.l, p.q) \tag{A.4}$$

The remaining integrals can be done similarly to get the result (4.17).

B Expansion of Beta Functions

The following expansion is useful:

$$\frac{\Gamma(1+\mu)\Gamma(1+\nu)}{\Gamma(1+\mu+\nu)} \;=\; 1-\mu\nu\zeta(2) + higher\, order \tag{B.1}$$

Using this and the recursion relation for the Gamma function, one finds:

$$B(p.q,q.l) = \frac{1}{p.q} + \frac{1}{q.l} - (p.q + q.l)\zeta(2) \tag{B.2}$$

$$-B(1 - p.q - q.l, q.l) = -\frac{1}{q.l} - (p.q + q.l)\zeta(2) \tag{B.3}$$

$$-B(1 - p.q - q.l, p.q) = -\frac{1}{p.q} - (p.q + q.l)\zeta(2) \tag{B.4}$$

Adding, we see that the poles cancel and we get

$$-3(p.q + q.l)\zeta(2) \tag{B.5}$$

C iϵ Prescription for Greens Function

The momentum representation (with some normalization)for the Greens function in two dimensions is:

$$G(\tau,\tau') \;=\; 1/2\int_{-\infty}^{+\infty} dp_0 \frac{1}{\sqrt{p_0^2+m^2}} e^{ip_0(\tau-\tau')} \tag{C.1}$$

Here m is an infrared regulator. Thus

$$\frac{\partial}{\partial\tau}G(\tau,\tau') \;=\; 1/2\int_{-\infty}^{+\infty} dp_0 ip_0 \frac{1}{\sqrt{p_0^2+m^2}} e^{ip_0(\tau-\tau')} \tag{C.2}$$

In the limit $m \to 0$ this becomes:

$$= \; i/2\int_{0}^{\infty} dp_0 e^{ip_0[(\tau-\tau')+i\epsilon]} \;+\; i/2\int_{-\infty}^{0} dp_0(-1)e^{ip_0[(\tau-\tau')-i\epsilon]} \tag{C.3}$$

where we have added the $i\epsilon$ to ensure convrgence. Thus we get

$$\frac{-i}{2}\left[\frac{1}{\tau-\tau'+i\epsilon} + \frac{1}{\tau-\tau'-i\epsilon}\right], \tag{C.4}$$

which is the Principal Value prescription. Note also from (C.2) that as $p \to 0$, $\partial_\tau G(\tau,\tau') \to 0$ as long as $m \neq 0$.

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